Modern approaches to quantum gravity

Homework 1 Fall 2025

1. The variational principle of General Relativity

In this exercise, we will show that the variational principle of Einstein gravity is ill-defined, and we will prove the need to add a boundary term to the action.

(a) First, remind yourself that the variation of the Einstein-Hilbert action can be written as

$$\delta S_{\text{EH}} = \underbrace{\frac{1}{16\pi G}\int d^d x\sqrt{-g}\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\Big)\delta g^{\mu\nu}}_{\text{e.o.m.}} + \frac{1}{16\pi G}\int d^d x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\,. \tag{1}$$

(b) Arguing that[1]

$$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\lambda\mu}\,, \tag{2}$$

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\nabla_\mu\delta g_{\nu\sigma} + \nabla_\nu\delta g_{\mu\sigma} - \nabla_\sigma\delta g_{\mu\nu})\,, \tag{3}$$

show that the variation becomes

$$\begin{aligned}\delta S_{\text{EH}} &= \text{e.o.m.} + \frac{1}{16\pi G}\int d^d x\sqrt{-g}\nabla_\lambda(g^{\lambda\sigma}\nabla^\mu\delta g_{\mu\sigma} - g^{\mu\nu}\nabla^\lambda\delta g_{\mu\nu}) \\ &= \text{e.o.m.} + \frac{\epsilon}{16\pi G}\int d^{d-1}x\sqrt{|\gamma|}n^\lambda(g^{\mu\nu}\nabla_\nu\delta g_{\mu\lambda} - g^{\mu\nu}\nabla_\lambda\delta g_{\mu\nu})\,,\end{aligned} \tag{4}$$

where $\gamma_{\mu\nu} = g_{\mu\nu} - \epsilon n_\mu n_\nu$ is the induced metric on the boundary and n^μ is an outward-point unit normal vector, $n^\mu n_\mu = \epsilon$, where $\epsilon = 1$ for time-like boundaries and $\epsilon = -1$ for space-like boundaries.

(c) We will use Dirichlet boundary conditions, by fixing $\delta g_{\mu\nu} = 0$ on the boundary. Thus derivatives tangent to the boundary are set to zero, $\gamma^{\mu\nu}\nabla_\nu\delta g_{\alpha\beta} = 0$. Prove that, with these boundary conditions,

$$\delta S_{\text{EH}} = \text{e.o.m.} - \frac{\epsilon}{16\pi G}\int d^{d-1}x\sqrt{|\gamma|}n^\lambda\gamma^{\mu\nu}\nabla_\lambda\delta g_{\mu\nu}\,. \tag{5}$$

This shows that Dirichlet boundary conditions are not sufficient and the variational principle is ill-defined. We will now investigated how to cure it.

(d) Let us define the *extrinsic curvature* as the Lie derivative of the induced metric along the normal vector,

$$K_{\mu\nu} \equiv \frac{1}{2}\mathcal{L}_n\gamma_{\mu\nu} = \frac{1}{2}(n^\rho\nabla_\rho\gamma_{\mu\nu} + \nabla_\mu n^\rho\gamma_{\rho\nu} + \nabla_\nu n^\rho\gamma_{\mu\rho})\,. \tag{6}$$

Arguing that $n^\mu\nabla_\alpha n_\mu = 0$, show that

$$K_{\mu\nu} = \frac{1}{2}(\nabla_\mu n_\nu - \epsilon n_\rho n_\mu\nabla^\rho n_\nu) + (\mu \leftrightarrow \nu)\,. \tag{7}$$

[1] You may use that $\delta R_{\mu\nu}$ and $\delta\Gamma^\rho_{\mu\nu}$ are tensors.

(e) Since n^μ is normal to a boundary, which can be defined by an equation $f(x^\mu) = 0$, one can write $n^\mu = \alpha(x)\nabla^\mu f(x)$. Using this, prove that

$$(\nabla_\mu n_\nu - \nabla_\nu n_\mu) + \epsilon(n^\lambda n_\nu \nabla_\lambda n_\mu - n^\lambda n_\mu \nabla_\lambda n_\nu) = 0\,, \tag{8}$$

and thus,

$$K_{\mu\nu} = \nabla_\mu n_\nu - \epsilon n_\rho n_\mu \nabla^\rho n_\nu\,. \tag{9}$$

(f) Defining

$$K = \gamma^{\mu\nu} K_{\mu\nu} = g^{\mu\nu} K_{\mu\nu}\,, \tag{10}$$

show that

$$K = \gamma^\nu{}_\mu \nabla_\nu n^\mu\,, \tag{11}$$

and that when the metric is held fixed on the boundary (Dirichlet boundary conditions),

$$\delta K = \gamma^\nu{}_\mu \delta\Gamma^\mu_{\nu\rho} n^\rho = \frac{1}{2} n^\lambda \gamma^{\mu\nu} \nabla_\lambda \delta g_{\mu\nu}\,. \tag{12}$$

Notice that this is exactly the boundary term we obtained in (5).

(g) Conclude by showing that

$$S = S_{\text{EH}} + \frac{\epsilon}{8\pi G} \int d^{d-1}x \sqrt{|\gamma|} K \tag{13}$$

has a well-defined variational principle with Dirichlet boundary conditions. This term is the so-called Gibbons-Hawking-York term.

2. Spacetime energy

In this exercise, we use two formalisms to compute the spacetime energy, the so-called ADM energy and the Komar mass.

(a) The ADM energy can be found as

$$E_{ADM} = -\frac{1}{8\pi G} \lim_{r\to\infty} \int_{S_r^2} d^2x \sqrt{\sigma}(K - K^0) \tag{14}$$

where K is the extrinsic curvature (11) of the 2-sphere of radius r (S_r^2) in curved space whereas K^0 is the extrinsic curvature of S_r^2 in flat space. Consider the Schwarzschild's black hole solution

$$ds^2 = -\Big(1 - \frac{2MG}{r}\Big)dt^2 + \frac{dr^2}{\Big(1 - \frac{2MG}{r}\Big)} + r^2 d\Omega^2 \tag{15}$$

Find the ADM energy of this spacetime on spatial hypersurfaces $\Sigma_t :\ t = \text{const}$.

(b) Now we will derive another formula to compute spacetime energy. To begin with, let us develop a formalism to compute Noether currents. For simplicity, consider a scalar field ϕ in d dimensions governed by an action $S(\phi)$. Firstly, note that under an *arbitrary* infinitesimal field variation $\phi \to \phi + \delta\phi$, the variation of the action can be written as

$$\delta S = \int d^d x (E\delta\phi + \partial_\mu(\Theta^\mu(\delta\phi))) \tag{16}$$

where E is the equation of motion. Secondly, note that if we have a symmetry with parameter ϵ, the action is invariant under some infinitesimal transformation $\phi \to \phi + \delta_\epsilon\phi$. Thus, it can be written as a total derivative,

$$\delta_\epsilon S = \int d^d x \partial_\mu M^\mu(\epsilon) \tag{17}$$

Equating the two equations in the case $\delta = \delta_\epsilon$, argue that

$$E\delta_\epsilon\phi = \partial_\mu M^\mu - \partial_\mu \Theta^\mu \tag{18}$$

The term Θ^μ is the so-called *presymplectic potential.* Argue that then, the current

$$J^\mu \equiv \Theta^\mu - M^\mu \tag{19}$$

is conserved on-shell.

(c) Now, we will compute the lower-degree Noether current associated to Einstein gravity by applying our general formalism. Let us therefore compute the various ingredients for the conserved charge.

Variation of the Lagrangian. Show that under a diffeomorphism, the Lagrangian[2] changes as

$$\delta_\xi(L\sqrt{g}) = \partial_\mu(\xi^\mu L\sqrt{g})\,. \tag{20}$$

[2]Note that L is now a scalar, rather than a tensor density.

Some ingredients you will need: a) the Lie derivative of a scalar; b) the Lie derivative of the metric; c) the fact that[3]

$$\delta\sqrt{g} = \tfrac{1}{2} g^{\mu\nu}\delta g_{\mu\nu}\sqrt{g}\,, \tag{21}$$

as well as the fact that for a vector,

$$\sqrt{g}\nabla_\mu V^\mu = \partial_\mu(\sqrt{g}V^\mu)\,, \tag{22}$$

(d) Remembering that

$$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\lambda\mu}\,,$$
$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\nabla_\mu \delta g_{\nu\sigma} + \nabla_\nu \delta g_{\mu\sigma} - \nabla_\sigma \delta g_{\mu\nu})\,, \tag{23}$$

show that the Einstein contribution to the symplectic potential is

$$\Theta^\mu(\delta g_{\nu\rho}) = \frac{\sqrt{g}}{16\pi G}\left(g^{\mu\nu}\nabla^\rho \delta g_{\nu\rho} - g^{\nu\rho}\nabla^\mu \delta g_{\nu\rho}\right). \tag{24}$$

(e) Combine the previous results to show that the expression for the Noether current for Einstein gravity is

$$J^\mu = \frac{\sqrt{g}}{16\pi G}\Big[\nabla_\nu(\nabla^\nu \xi^\mu - \nabla^\mu \xi^\nu) + 2G^{\mu\nu}\xi_\nu\Big]\,. \tag{25}$$

Note that the last term vanishes on-shell, by the Einstein equations of motion. (Hint: You will need to use the definition of the Riemann tensor in an intermediate step.)

(f) Now consider a spatial slice Σ. Let h_{ab} be the induced metric on Σ and n^a is a unit vector normal to Σ.
Let σ_{ab} be the induced metric on the spatial boundary of Σ, $\partial\Sigma$, and σ^a is a unit vector normal to $\partial\Sigma$.
Defining the rescaled (on-shell) Noether current as

$$j^\mu \equiv \frac{1}{8\pi G}\nabla_\nu(\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu)\,, \tag{26}$$

show that the associated conserved quantity can be expressed as a boundary integral over $\partial\Sigma$

$$E_k[\Sigma] \equiv \int_\Sigma \sqrt{h}\, n^a j_a = \frac{1}{4\pi G}\int_{\partial\Sigma} \sqrt{\sigma}\, n^a \sigma^b \nabla_a \xi_b \tag{27}$$

This is called the *Komar mass* of the spacetime. Argue why this quantity can be interpreted as the energy.

(g) Verify that $E_k[\Sigma_t] = E_{ADM}$ for Schwarzschild's black hole using $\xi = \partial_t$.

[3] See e.g. Sean Carroll's GR book, pg. 163.